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Fuel Sloshing in a Spherical Tank Filled to an Arbitrary Depth

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The kernel function for liquid sloshing in a spherical tank filled to an arbitrary depth is shown to be related to Green's function of the second kind and is constructed successfully by numerical means. Natural frequencies are then computed as eigenvalues of a matrix. Eigenfunctions are obtained, at a finite number of points, as the eigenvectors that are sufficient for approximate evaluation of the force acting on the container. Simple formulas of force and moment are given for both pitching and translational oscillation under a fixed gravitational field. Finally, comparisons of both predicted natural frequencies and force response with the experimental results for a quarter-full tank are given.

Nomenclature

a = radius of the spherical tank
A_F = area of undisturbed free surface
A_R = area of wetted surface of sphere
b = maximum value of ρ, radius of free surface
d = tank diameter, $2a$
C_{ij} = g_{ij}
$D(q)$ = $[E(q) - K(q)]/q^2$
$D(\beta, k')$ = $[E(\beta, k') - k' \sin\beta]$
$DKEF$ = a function in the computer program (cf. Appendix C)
F = undisturbed free surface
F_s = horizontal force acting on the tank due to fuel sloshing
F_{ij} = $(\rho_i\rho_j)^{1/2}F_0(\rho_j, \rho_i) = F_{ji}^{(0)}$ (note the order of ρ_i, ρ_j in F_{ij})
$F_0(\rho, \rho')$ = integrated kernel function related to $\partial G_0/\partial n$
g = effective gravitational acceleration
g_{ij} = $(\rho_i\rho_j)^{1/2}h_1(\rho_i, \rho_j)$
$G(P, Q)$ = Green's function of the second kind for the spherical bowl
$G_0(P, Q)$ = Green's function of the second kind for a sphere
$h(P, Q)$ = additional part of Green's function for spherical tank other than half-full
$h_1(\rho, \rho')$ = integrated kernel function related to $h(P, Q)$
$H(\rho, \rho')$ = integrated kernel function related to $G(P, Q)$
$H_0(\rho, \rho')$ = integrated kernel function related to $G_0(P, Q)$
H_{ij} = $(\rho_i\rho_j)^{1/2}H(\rho_i, \rho_j)$
$H_{ij}^{(0)}$ = $(\rho_i\rho_j)^{1/2}H_0(\rho_i, \rho_j)$
I = point of integration, except $[I]$ being the unit matrix
$K(q)$, $E(q)$ = complete elliptic integrals of first and second kind, respectively
M_L = total mass of liquid (fuel)
n = outer normal
$NEFF$ = a function in the computer program (cf. Appendix C)
$P(Q)$ = $[K(q) - (\pi/2)]$ (cf. Appendix C)
$\bar{P}(r, \psi)$ = a ring corresponding to $P(r, \psi, \theta)$
q_1 = $2(\rho\rho')^{1/2}/(\rho + \rho')$
q_2 = $2a(\rho\rho')^{1/2}/[(\rho\rho' - b^2)^2 + z_F^2(\rho - \rho')^2 + 4\rho\rho'a^2]^{1/2}$
$\bar{Q}$, $\bar{I}$ = analogous to P but related to Q and I, respectively
r, ψ, θ = spherical coordinates
R = the wetted spherical surface before sloshing unless defined by (A2)
$R'(P, P')$ = $[r'^2 + (a^4/r^2) - 2(r'/r)a^2\cos\gamma]^{1/2}$
R_{PQ} = distance between the points P and Q
dS = element of surface
$d\bar{S}$ = $dS/d\theta \to \rho d\rho$ on F
U = horizontal displacement of container in the x direction
x = $r\cos\theta$
z_F = vertical distance of free surface from center of sphere; positive upward
α_n = defined by Eq. (11b) $\int_F \phi_n^2(I)dS_I$
β_n = defined by Eq. (11a)
$\cos\gamma$ = angle between the vectors **OP** and **OP'**
λ_1, λ_2 = $(1/\rho\rho')\{-z_F^2 \pm [z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2\rho'^2]^{1/2}\}$, respectively
$\Lambda_0(\psi, q)$ = Heuman's lambda function (Ref. 9)
$\tilde{\lambda}_n$ = ω_n^2/g
$\Pi(\alpha^2, q)$ = complete elliptic integrals of the third kind (Ref. 9)
ρ = radial distance from a point on the free surface to the center of the free surface
ρ', ρ_j = ρ of integration variable
ρ_L = density of liquid (fuel)
ϕ_v = velocity potential, $\nabla\phi_k = \mathbf{q}$, $\mathbf{q}$ being the velocity vector

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ϕ_n = nth eigenfunction
Φ_n = nth integrated eigenfunction related to ϕ_n
$\Phi(n)$ = $\rho^{1/2}\Phi_n(\rho)$
ω = frequency of oscillation
ω_n = nth resonant frequency
Ω_n = $\omega_n^2 a/g$, nth resonant frequency parameter

Subscripts

F = related to surface F
i, j, k = related to ρ_i, ρ_j, ρ_k, respectively
I = related to integration variables
P = related to the point $P(r, \psi, \theta)$ or $P(\rho_P)$
Q = related to the point $Q(r, \psi, \theta)$ or $Q(\rho_Q)$
R = related to surface R

Introduction

DISTURBANCES on a rocket or missile can induce sloshing of fuel in a partially filled tank. Sloshing in turn exerts excitation forces on a vehicle that, in some cases, can be detrimental to the trajectory or even result in loss of control. Sloshing in a circular cylindrical tank has been widely investigated with and without damping. To facilitate dynamic analysis, an equivalent mechanical model for a circular tank is given in Ref. 3. For a spherical tank, an ingenious seminumerical method is given in Ref. 4. However, the solutions of the problem are restricted to three special cases, namely, nearly full, nearly empty, and half-full tanks. The restriction is due to the lack of Green's function of the second kind (Neumann function) for the spherical bowl. Although Green's function of the first kind for the spherical bowl is given in Ref. 5, it is doubtful that a simple expression for Green's function of the second kind exists in the toroidal coordinates since the normal derivative on the spherical cap is a combination of two derivatives in this coordinate system. The sequence method given in Ref. 6 is convergent for Green's function of the first kind but may diverge for the second kind. One may resort to the Liouville-Neumann method (series method[7]) and prove it converges, but, when Green's function on the boundary is desired, the kernel function is singular; thus the series becomes increasingly more difficult to evaluate when more terms are needed. If we do not employ the Neumann function, an integral equation on the free surface is also obtained. Unfortunately, the eigenfunctions no longer satisfy the necessary orthogonal relationship[8]; thus they are the desired eigenfunctions only if the Neumann function is employed.[4] In this paper, a a numerical scheme is devised to determine the desired kernel function, which is one component of the Neumann function, and then the same procedure as given in Ref. 4 is applied to evaluate the sloshing characteristics. Considerably more work is required to calculate the pressure on the wall using this approach, although, in principal, this can be done.

After the theory in this paper was developed, some other approaches were published. One approach[9] seeks the variational solution based on Hamilton's principle through the Rayleigh-Ritz method.† Since only an integrated free surface condition was imposed, it is somewhat doubtful that an accurate prediction of force response or pressure can be assured,[10] although error in the lowest mode frequency was less than 1% for a flat cylindrical tank. In another approach,[11] finite difference techniques were employed to seek eigenvalues in a boundary condition by three different methods. Method I and method III[11] use either an iterative approach to calculate the Rayleigh quotient or a Rayleigh-Ritz procedure to minimize the Rayleigh quotient, but are somewhat inferior[11] to the Rayleigh-Ritz procedure applied to the continuous domain without employing finite differences. Method II[11] converts the problem into an equivalent matrix eigenvalue problem by eliminating the points outside the free surface through an inversion of the matrix if the number of other points is small or through an influence coefficient-type calculation if otherwise. In the latter case, if there are N points on the free surface, N boundary-value problems should be solved first (say, by successive overrelaxation) before reduction to the eigenvalue problem of an $N \times N$ matrix. Depending to a large extent on the number of net points required for a desired accuracy (say, three figures in frequencies and force response), the computing time (based on estimation on a GE-225 computer)‡ of the last method for a spherical tank seems to be comparable to the present method. Although further (significant) acceleration of the rate of convergence of the subroutines in the present method in the present problem may be quite difficult, an alternative numerical scheme has been devised which is expected to reduce the computing time to one-half or further.§ Finally, Ref. 12 has also been published in which the kernel function is constructed empirically, based on knowledge for a half-full and full tank.

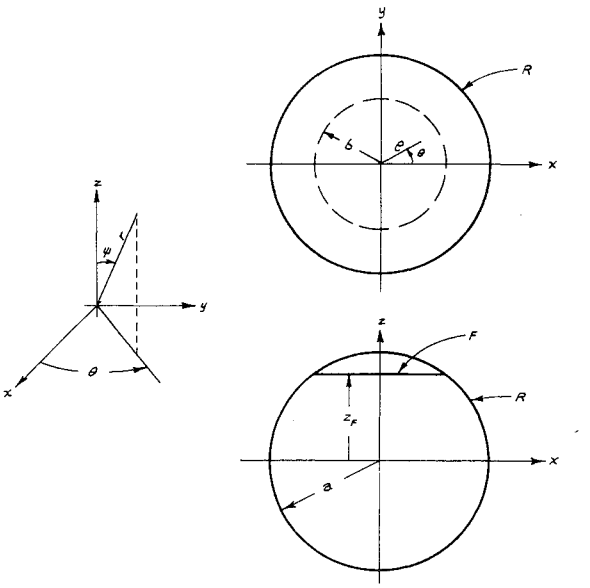

Fig. 1 Graphical illustration of some nomenclatures.

The purpose of this paper is not only to predict the natural frequencies and force response of liquid sloshing but also to show how kernel functions are related to the Neumann function on the boundary and how they can be constructed numerically for a spherical tank. Analogous extensions to other configurations or other problems may be possible but will not be treated in this paper.

Mathematical Formulation

A. Kernel Function

Let $G(P, Q)$ and $G_0(P, Q)$ be Green's function of the second kind for the interior of the given spherical bowl (Fig. 1) and the sphere, respectively. Then 1) both $G(P, Q)$ and $G_0(P, Q)$ possess continuous second derivatives and satisfy the Laplace equation inside the bowl and the sphere, respectively, except at the point $P = Q$; 2) both G and G_0 possess a unit

† This method has been applied to a spherical tank by Riley and Trembath whose results are compared with the present for a one-fourth full tank in Fig. 6 of Refs. 1 and 2.

‡ It is estimated under the assumption that there are 20 free surface points and 300 total net points with 120 iterations for each boundary-value problem (based on experience of a similar problem) and average speed for 5 multiplications, 4 additions, and 1 additional multiplication or division at each point in each iteration. There are other estimates based on experiences which yield approximately the same magnitude of computing time.

§ Detailed description of the second numerical scheme is given in Refs. 1 and 2.

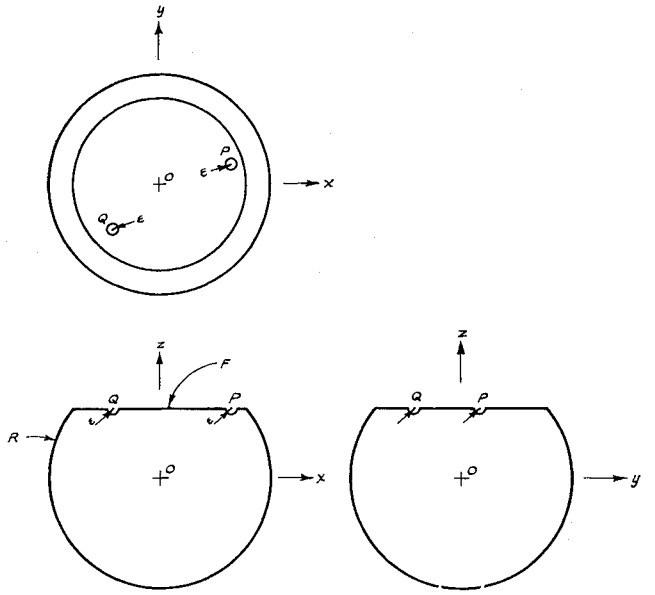

Fig. 2 Surface of integration.

sink, $1/4\pi R_{PQ}$ at $P = Q$ inside the bowl; 3) $\partial G_0/\partial n = -(1/4\pi a^2)$ on the whole surface of the sphere, $\partial G/\partial n = k_1 = -1/(A_R + A_F)$ on the surfaces of the bowl, R and F; 4) G_0 be that given in Ref. 9; and 5) $G(P, Q)$ satisfies the normalizing condition[13]

$$\int_{R+F} G(I, Q)dS_I = 0$$

Following these conditions, it is well known[13] that the Neumann function G is symmetric as well as G_0, i.e., $G(P, Q) = G(Q, P)$, $G_0(P, Q) = G_0(Q, P)$. When P, Q are both interior points, analogous to the proof of symmetric properties, one has

$$G(Q, P) - G_0(P, Q) =$$

$$\int_{R+F} \left\{ G_0(I, Q)\frac{\partial G(I,P)}{\partial n_I} - G(I, P)\frac{\partial G_0(I, Q)}{\partial n_I} \right\} dS_I =$$

$$k_1 \int_{R+F} G_0(I, Q)dS_I - \int_F G(I, P)\cdot\left[\frac{\partial G_0(I, Q)}{\partial n_I} + \frac{1}{4\pi a^2}\right] dS_I \quad (1)$$

which is an integral equation governing $G(P, Q)$, where P, Q are inside the bowl, not on R and F.

For values of Green's function with P, $Q (P \neq Q)$ both on F, not on R, apply directly the divergence theorem to the surface shown in Fig. 2. Since there is an infinitesimal semi-sphere around the sinks at P and Q, respectively, one finds

$$\frac{1}{2} G(Q, P) - \frac{1}{2} G_0(P, Q) =$$

$$\int_{\substack{R+F \\ P\neq I\neq Q}} \left[G_0(I, Q)\frac{\partial G}{\partial n_I} - G(I, P)\frac{\partial G_0(I, Q)}{\partial n_I} \right] dS_I \quad (2)$$

By making P and Q in Eq. (1) approach P and Q on the free surface along its normal, Eq. (1) can be reduced to Eq. (2).

In Ref. 4, for fuel sloshing in a spherical tank, only those eigenfunctions proportional to $\cos\theta$ are needed. One shall see in the next section that it is sufficient to know one component $H(\bar{P}, \bar{Q})$ of the Neumann function $G(P, Q)$ to determine the sloshing characteristics. Let

$$\begin{Bmatrix} H(\bar{P}, \bar{Q}) \\ H_0(\bar{P}, \bar{Q}) \end{Bmatrix} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} \begin{Bmatrix} G(P, Q) \\ G_0(P, Q) \end{Bmatrix} \cos\theta_P \cos\theta_Q d\theta_P d\theta_Q \quad (3a)$$

$$h_1(\bar{P}, \bar{Q}) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} [G(P, Q) - G_0(P, Q)] \times \cos\theta_P \cos\theta_Q d\theta_P d\theta_Q = H(\bar{P}, \bar{Q}) - H_0(\bar{P}, \bar{Q}) \quad (3b)$$

Since G and G_0 are symmetric functions, H, H_0, and thus h_1 are symmetric functions.

For points $\bar{P}$, $\bar{Q}$ corresponding to P and Q, respectively, inside the spherical bowl, Eq. (1) can be integrated to yield

$$h_1(\bar{P}, \bar{Q}) = -\int_{\bar{F}} F_0(\bar{I}, \bar{Q})H_0(\bar{P}, \bar{I})d\bar{S}_I - \int_{\bar{F}} F_0(\bar{I}, \bar{Q})h_1(\bar{P}, \bar{I})d\bar{S}_I \quad (4)$$

for which the reversing of orders of integration are applied and can be justified by carrying out the details. The function F_0 is defined by

$$F_0(\bar{I}, \bar{Q}) = \frac{1}{\cos\theta_I}\int_0^{2\pi} \frac{\partial G_0(I, Q)}{\partial n_I} \cos\theta_Q d\theta_Q = \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} \cdot \frac{\partial G_0[r_I, r_Q;\ \psi_I, \psi_Q;\ \cos(\theta_I - \theta_Q)]}{\partial n_I} \times \cos\theta_Q \cos\theta_I d\theta_Q d\theta_I \quad (5)$$

which is a nonsymmetric function as $\partial G_0/\partial n_I$.

Similarly, if both P and Q are on F, not on R, integration of Eq. (2) yields

$$\frac{1}{2}(\rho_P\rho_Q)^{1/2}h_1(\rho_P, \rho_Q) = -\int_0^b [(\rho_I\rho_Q)^{1/2}F_0(\rho_I, \rho_Q)][(\rho_I\rho_P)^{1/2}H_0(\rho_P, \rho_I)]d\rho_I - \int_0^b [(\rho_I\rho_Q)^{1/2}F_0(\rho_I, \rho_Q)][(\rho_I\rho_P)^{1/2}h_1(\rho_P, \rho_I)]d\rho_I \quad (6)$$

where F_0 and H_0 are given in Appendixes A and B. For $z_F = 0$, $F_0 = 0$ almost everywhere on F; hence, for a half-sphere $h_1(P, P') = 0$, $H = H_0$, which is in agreement with Ref. 4.

B. Eigenfunctions

The eigenfunctions ϕ_n's are assumed to possess the following properties: 1) ϕ_n is regular inside the bowl and $\nabla^2\phi_n = 0$, 2) $\partial\phi_n/\partial n_I = 0$ on R, $(\partial\phi_n/\partial n_I) = \check{\lambda}_n\phi_n(I)$ on F, and 3) $\phi_n(P) = \Phi_n(\bar{P})\cos\theta_P$.[4] The last condition is appropriate for translational oscillation of the tank. Analogous to Eqs. (1) and (2),

$$\phi_n(P) = \int_F G(I, P)\check{\lambda}_n\phi_n(I)\, dS_I \quad (7)$$

when P is inside the spherical bowl $R + F$ and

$$\frac{1}{2}\phi_n(P) = \int_F G(I, P)\check{\lambda}_n\phi_n(I)dS_I \quad (8)$$

when P is on F.

Analogous to Eqs. (4) and (5) by integration,

$$\Phi_n(\bar{P}) = \int_F H(\bar{I}, \bar{P})\Phi_n(I)d\bar{S}_I \qquad \bar{P} \text{ not on } \bar{F} \quad (9)$$

$$\frac{1}{2}\Phi_n(\rho) = \check{\lambda}_n\int_0^b H(\rho, \rho')\Phi_n(\rho')\rho' d\rho' \qquad \bar{P} \text{ on } \bar{F} \quad (10)$$

This shows only $H(\bar{P}, \bar{Q})$ is needed for the pertinent eigenfunctions.

C. Sloshing Force and Pressure (Translational Oscillation)

By introducing a displacement potential relative to the tank $\Psi_d = \Sigma a_n(t)\phi_n(r, \psi, \theta)$, the sloshing force acting on the

container is derived from the Lagrangian equation in Ref. 4, namely,

$$F_s = -M_L\ddot{U} - \rho_L\Sigma\beta_n\ddot{a}_n \tag{11}$$

where M_L is the mass of the liquid, equal to $\rho_L(\pi/3)[2a^3 + 3a^2z_F - z_F{}^3]$,

$$\beta_n = \frac{\omega_n{}^2}{g}\int_F x\phi_n dS = \pi b^2\Omega_n\frac{b}{a}\int_0^b \Phi_n(\rho_I)\rho_I{}^2 d\rho_I \tag{11a}$$

$$\alpha_n = \int_F \phi_n{}^2 dS = \pi b^2\int_0^b \Phi_n{}^2(\rho_I)\rho_I d\rho_I \tag{11b}$$

$$\ddot{a}_n = \frac{(g/\omega_n{}^2)\cdot(\beta_n/\alpha_n)\ddot{U}}{(\omega_n{}^2/\omega^2) - 1} \tag{11c}$$

The velocity potential

$$\phi_v = \sum_{n=1}^{\infty}\dot{a}_n(t)\phi_n(r, \psi, \theta) + \dot{U}x \tag{12}$$

The pressure on the container

$$p \cong \rho\Sigma\ddot{a}_n(t)\phi_n(a, \psi, \theta) - \rho\ddot{U}x \cong -\rho_L(\partial\phi_v/\partial t) \tag{13}$$

within the accuracy of the linearized theory. Equation (11) can also be obtained easily by integration of pressure.[1, 2]

Once ϕ_n on F is evaluated, one may employ G_0 to obtain $\phi_n(P)$ from

$$\phi_n(P) = \int_{R+F}\left[G_0(I, P)\frac{\partial\phi_n}{\partial n_I} - \phi_n(I)\frac{\partial G_0(I, P)}{\partial n_I}\right]dS_I = \Phi_n(\bar{P})\cos\theta_P \tag{14a}$$

$$\Phi_n(\bar{P}) = \int_F H_0(\bar{I}, \bar{P})\lambda_n\Phi_n(I)dS_I - \int_{\bar{F}} F_0(\bar{I}, \bar{P})\Phi_n(I)dS_I \tag{14b}$$

The integral on R dropped out as $\partial\phi_n/\partial n_I = 0$ on R and $\partial G_0/\partial n_I = \text{const}$ on R.

For P on R, not on F, the integrands of the integrals in Eq. (14b) are nonsingular; hence, $\Phi_n(\bar{P})$ can be calculated by well-known numerical methods. For contact points both on R and F, the value of $\Phi_n(\rho_n)$ may be obtained by evaluation of the integral by midpoint formula.

D. The Moment of the Tank

Since the moment about the center of the sphere is zero, the moment about any point on the z axis at a distance l below it is

$$M_l = F_s l \tag{15}$$

E. Pitching Oscillation

From the equivalence of boundary conditions, it can be shown (Refs. 1 or 2) that a small amplitude pitching oscillation is equivalent to a translational oscillation of amplitude, $l\theta y$, θy being the angular displacement and l being the same as that in Eq. (15). There is, however, an additional x force and an additional y moment due to static tipping pressure[14] or due to the weight of the liquid.

F. Mechanical Model

An equivalent mechanical model for sloshing in a spherical tank is given in Ref. 15, but unfortunately the extrapolation to include damping was not as successful as in the case of a cylindrical tank[3] and could only be used for order of magnitude estimates.[15]

Numerical Method

A. Approximate Determination of the Kernel Function at a Finite Number of Points

A numerical quadrature formula will be used in order to replace the integral equation (6) by a matrix equation. In doing so, a minor difficulty arises because of the presence of logarithmic singularity at $\bar{P} = \bar{I}$ or $\rho_i = \rho_j$, the latter of which is the integration variable. In the original manuscript, an attempt was made to devise a more sophisticated quadrature formula, expecting higher accuracy. Unfortunately, it seems to contain integrals that are difficult to express in known functions, or which require a very careful process of taking limits under the integral signs. Further, the apparent higher-order terms actually may be very large and not negligible. Therefore, to reduce the total effort, the present numerical scheme based on a generalized midpoint formula is devised.

The integrals are divided into N equal parts ($N = 20$ will be used), and the field point is one of the centers of the intervals. A simple midpoint formula will not be applicable when the logarithmic singularity¶ appears at the midpoint, but, if the interval is subdivided into four intervals (or more), the error may become acceptable. For example, consider the integral

$$S = \int_{\rho_i-(\Delta/2)}^{\rho_i+(\Delta/2)} \ln|\rho_i - \rho_j|d\rho_j = \Delta\ln\frac{\Delta}{2} - \Delta \cong \Delta\ln\Delta - 1.69315\Delta \tag{16}$$

With four equal subintervals, the midpoint formula yields

$$S = \left[2\ln\left(\frac{1}{8}\Delta\right) + 2\ln\left(\frac{3}{8}\Delta\right)\right]\frac{\Delta}{4} \cong \Delta\ln\Delta - 1.5301\Delta \tag{17}$$

The error is 0.16Δ. For $\Delta = \frac{1}{20}$ ($N = 20$, $b = 1$), the relative error is less than 0.18%. The integral equation (6) is therefore reduced approximately to the matrix equation, the unknown function at the four midpoints being replaced by its value at the center of the interval containing these four subintervals:

$$\tfrac{1}{2}[C] = -[M] - [C][D] \tag{18}$$

the solution of which can be easily found by matrix inversion.

In Eq. (19), $[C]$, $[M]$, and $[D]$ are square matrices of which the elements are given in Refs. 1 and 2. As a check of accuracy, the symmetric property $C_{ij} \cong C_{ji}$ should hold approximately true. Then we can use the average value for the corrective term in the kernel function.

The difficulty of the problem, however, lies in the accurate and rapid evaluation of the function $F_{ij}{}^0$ and $H_{ij}{}^0$ contained in the matrix elements of $[M]$ and $[D]$.

B. Determination of Eigenvectors, Natural Frequencies, and Force

The matrix approximation of Eq. (10) is

$$\left[\frac{N}{2\Omega_n(b/a)}\right]\{\Phi^{(n)}\} = [A]\{\Phi^{(n)}\} \tag{19}$$

where the factor $\frac{1}{2}$ on the right-hand side is in agreement with Ref. 1, since the strength of Green's function has not been doubled in this paper. The elements of the matrix A are given in Refs. 1 and 2. The largest few of the eigenvalues yield the lowest few of the natural frequency parameters. Presumably, as in other sloshing problems, a few eigenvectors are sufficient for numerical evaluation of the force response in the "neighborhood" of the first natural frequency.

C. Precision Problem

The functions F_0, H_0 have been first expressed in terms of complete elliptical integrals of the first kind, the second kind, and the third kind, and of simple elementary functions

¶ This is known from the analytic expression of H^0 and F given in Appendixes A and B.

Table 1 Samples of corrective part C_{ij} for quarter-full tank

i/j	1	2	9	10	11	19	20
1	1.0683901 $\times 10^{-5}$	5.419177 $\times 10^{-5}$	...	8.2274045 $\times 10^{-4}$	...	1.8775441 $\times 10^{-3}$	1.9692882 $\times 10^{-3}$
2	5.5416096 $\times 10^{-5}$	2.8747310 $\times 10^{-4}$	...	4.2729657 $\times 10^{-3}$	...	9.7666734 $\times 10^{-3}$	1.0244200 $\times 10^{-2}$
9	...	...	4.8220011 $\times 10^{-2}$	5.6585628 $\times 10^{-2}$	6.5237782 $\times 10^{-2}$	...	...
10	8.2212278 $\times 10^{-4}$	4.2698876 $\times 10^{-3}$	5.6576890 $\times 10^{-2}$	6.6504031 $\times 10^{-2}$	7.6811489 $\times 10^{-2}$	1.6376404 $\times 10^{-1}$	1.7254674 $\times 10^{-1}$
11	...	...	6.5216369 $\times 10^{-2}$	7.6797981 $\times 10^{-2}$	8.8875807 $\times 10^{-2}$	...	...
19	1.8737714 $\times 10^{-3}$	9.7465323 $\times 10^{-3}$	...	1.6341799 $\times 10^{-1}$	...	5.6357121 $\times 10^{-1}$	6.3742676 $\times 10^{-1}$
20	1.9743665 $\times 10^{-3}$	1.027057 $\times 10^{-3}$	...	1.7294743 $\times 10^{-1}$	...	6.3772697 $\times 10^{-1}$	7.6960677 $\times 10^{-1}$

(Appendixes A, B). The elliptical integrals of the third kind are expressed in terms of Heuman's lambda function Λ_0,[16] which is again expressed either in a series form or in a close form of incomplete and complete elliptical integrals of the first and second kind, i.e.,

$$\Lambda_0(\beta, k) = (2/\pi)[E(k)F(\beta, k') + K(k)E(\beta, k') - K(k)F(\beta, k')] \quad (20)$$

In F_3 and H_{03}, a serious precision problem occurs because of the almost complete loss of significant figures in subtractions for ρ_i, ρ_j both small. At first, the series form of the lambda function was used, but it was found that the series is very slowly convergent when the parameter is near unity, especially if double precision or twelve significant figures are sought. Then, the iterative methods for evaluating elliptic integrals[17] are used which converge to 10^{-9} within four or five iterations. Although the complete elliptic integrals can be computed very rapidly, the subroutine *NEFF* (see Appendix C) for incomplete integrals and differences related to them consumes 8 sec (used twice), whereas the total time for evaluating F_0, H_0 is only 25 sec at each point, all on the GE-225 computer. Longer time would be required for higher precision as the number of iterations increases.

To increase the precision, analytic subtractions are made so that no significant subtraction remains, if possible. Noniterative subtractions in which four or less figures are lost are acceptable if four or more significant figures out of eight (single regular precision on the machine) are desired. The technique can be illustrated by the following cases:

1) Let $(A - B)$, the difference of A and B, be small but expressible analytically without subtraction. Then, for example, $(1/A)^{1/2} - (1/B^{1/2})$ should be evaluated from $(-1/A^{1/2}B^{1/2})(A - B)/(A + B^{1/2})$, e.g., $A = 2, B = 2 + \delta$, $\delta \ll 1$, $(A - B) = \delta$.

2) Let the k_n's be small (positive) quantities containing no subtraction; then $(1 + k_1)(1 + k_2) \ldots (1 + k_n) - 1$ should be evaluated by repeated application of the simple relation that $(1 + k_1)(1 + k_2) - 1 = k_1 + k_2 + k_1k_2$.

3) To subtract a desired quantity from a known function may require a new subroutine for this function performing significant subtraction analytically, e.g., *NEFF* (Appendix C).

Aside from relatively mechanical operations, the device of *DKEF* and *NEFF* subroutines, a function Π_n was used. It is given in Appendix D.

It is noted that, after a small manipulation, direct numerical integration of the integrals F_3, H_{03} at sampling points of the entire domain of ρ_i, ρ_j was also computed by

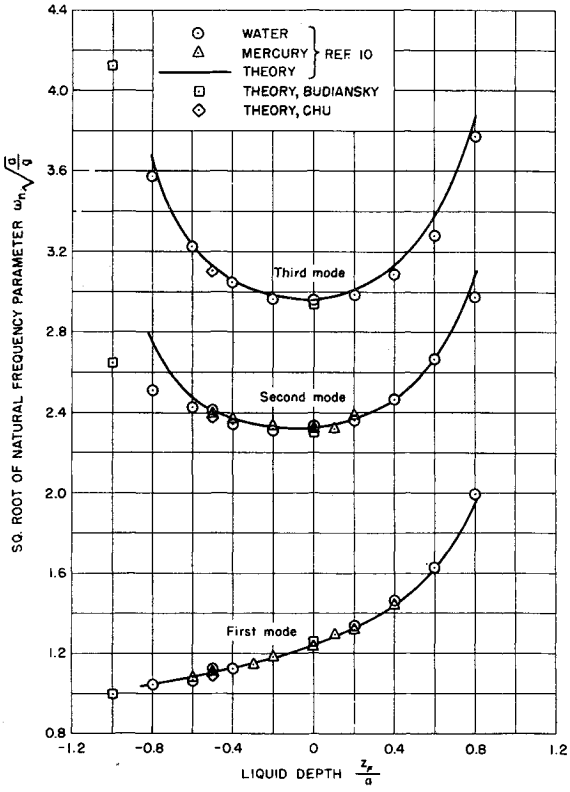

Fig. 3 Comparison of the first three natural frequencies with experiments of Stofan-Armstead.[12]

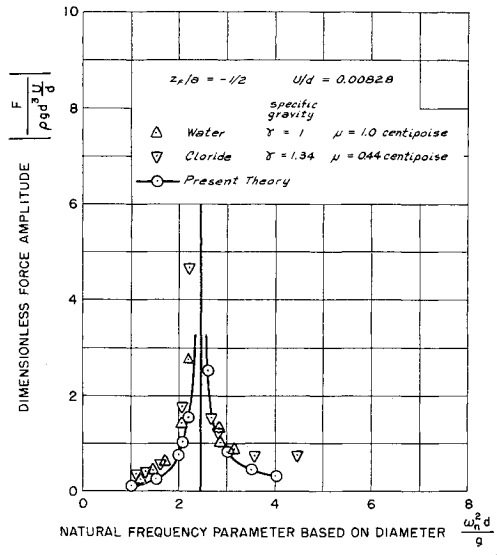

Fig. 4 Comparison of force response for quarter-full tank with experiments by Abramson et al.[15]

Weddle's rule. Although four or more significant figures can be obtained, it is deemed too slow over the major part of the domain. For instance, it took about 5 and $2\frac{1}{2}$ min, respectively, for F_3 and H_{03} on a GE-225 computer with 384 intervals, or a relative error of about 10^{-5} otherwise at a point near the right lower corner of the domain (ρ_i, ρ_j near unity). These values at sampling points are valuable as they serve as a good check on the present computer program that evaluates F_1, F_2, F_3, H_{03}, H_{02}, H_{03} all together at a rate of 25 sec/net point (on the same computer with an accuracy of four or more significant figures).

Example: Fuel Sloshing in a Quarter-Full Tank

First, F_0 and H_0 are generated, then the matrix equation (19) is solved. The corrective part C_{ij} to the kernel function obtained is symmetric almost to four figures (Table 1). The relative errors in the sample points are less than 0.3% or better. Since these values are quite representative, the values of C_{ij} at other points are not shown in the table.

Next, the eigenvalues and eigenvectors of Eq. (20) and then the force response of Eq. (11) are calculated. The calculated first four eigenvalues are 9.48863, 2.0591201, 1.2003387, 0.84773955, respectively. The corresponding frequency parameters are compared with experiments in Fig. 3. It seems that the values are well within possible experimental error, although it may be slightly less than the actual value, noting that natural frequencies are somewhat smaller for large amplitudes of oscillation.

The constants needed to calculate the force response are compared with graphical values given by Ref. 4 in Table 2. Since the coefficient D_1^2/C_1 is in agreement with Budiansky's value, the main difference lies in the value of first natural frequency for frequency range in its neighborhood. Since the graphically interpolated value is less reliable, which is also confirmed experimentally in this case, only the present theory is compared with experiments[15] in Fig. 4. The difference between theory and experiments, perhaps, is essentially due to finite amplitude effect, but the agreement seems to be quite reasonable.

Conclusions and Discussions

The present theory and computer program seem to yield satisfactory predictions of natural frequencies and force response in comparison with experiments for a quarter-full spherical tank. The computer program is expected to be applicable to other liquid depths, although not beyond improvement in efficiency. The results also confirm the theory that the kernel function is related to the Neumann function on the boundary and that this function can be constructed by adding a corrective part to a known Green's function numerically for practical applications. Extensions to other problems may be possible, but one must resolve the precision problem if it exists. One may also find a more sophisticated numerical scheme to be more desirable, either in accuracy or in efficiency.

Table 2 Comparison of constants with data from Ref. 4

	$\Omega_n = \omega_n^2 a/g$		$D_n^2/C_n = [\beta_n/\Omega_n b/a)]^2 \alpha_n{}^*$	
n	Chu	Budiansky, Fig. 9, Ref. 1	Chu	Budiansky, Fig. 10, Ref. 1
1	1.2169314	1.13	0.24919	0.250
2	5.6077376	5.44	0.44687×10^{-3}	0.417×10^{-3}
3	9.6197893	9.23	0.44758×10^{-4}	0.766×10^{-4}
4	13.620935	...	0.112633×10^{-4}	...

Appendix A: Analytic Expression for $H_0(\rho, \rho')$

Green's function of the second kind for a whole sphere[18] is

$$G_0(P, P') = \frac{1}{4\pi}\left\{\frac{1}{R} + \frac{a}{rR'} + \frac{1}{a}\ln\frac{2a^2}{a^2 - rr'\cos\gamma + rR'}\right\} \tag{A1}$$

where

$$R = (r^2 + r'^2 - 2rr'\cos\gamma)^{1/2} \tag{A2}$$

$$R' = \left(r'^2 + \frac{a^4}{r^2} - \frac{2r'}{r}a^2\cos\Delta\right)^{1/2}$$

$$\cos\gamma = \cos\psi\cos\psi' + \sin\psi\sin\psi'\cos(\theta - \theta') \tag{A3}$$

Using Eq. (A1), with $\sigma = \theta - \theta'$ and $z = z' = z_F$,

$$\begin{aligned} H_0(\bar{P}, \bar{P}') &= \frac{1}{\cos\theta}\int_0^{2\pi} G_0(P, P')\cos\theta' d\theta' \\ &= \int_0^{2\pi} G_0(r, r';\ \cos\sigma;\ \psi, \psi')\cos\sigma d\sigma \\ &= H_{01} + H_{02} + H_{03} \end{aligned} \tag{A4}$$

where

$$H_{01} = \frac{1}{2\pi(\rho\rho')^{1/2}}q_1[-K(q_1) - 2D(q_1)] \tag{A5}$$

$$H_{02} = \frac{q_2}{2\pi(\rho\rho')^{1/2}}\{-K(q_2) - 2D(q_2)\} \tag{A6}$$

From algebraic manipulations and integration by parts,

$$\begin{aligned} H_{03}(\rho, \rho') = \frac{1}{4\pi}\int_0^{2\pi}\frac{1}{a}\cdot \ln\left[\frac{2a^2}{a^2 - z_F^2 - \rho\rho'\cos\sigma + [(\rho\rho' - b^2)^2 + 2\rho\rho' a^2(1 - \cos\sigma) + z_F^2(\rho - \rho')^2]^{1/2}}\right]\cdot\cos\sigma d\sigma = \\ \frac{1}{2\pi a}\left\{\left[\frac{1}{\rho\rho'}\left(\frac{2(\rho\rho')^{1/2}}{q_2}\right)E(q_2)\right] + \frac{1}{\rho\rho'}\left(\frac{q_2}{2a(\rho\rho')^{1/2}}\right)\cdot[\rho^2\rho'^2 + b^4 + z_F^2(\rho^2 + \rho'^2) - 2a^2b^2]K(q_2) - 2\left(\frac{q_2}{2a(\rho\rho')^{1/2}}\right)\cdot \right. \\ \left[\frac{1}{\rho\rho'}[\rho^2\rho'^2 + b^4 + z_F^2(\rho^2 + \rho'^2) - a^2b^2]\right]\left[\frac{\lambda_1 - 1}{\lambda_1 - \lambda_2}\Pi\left(\frac{2}{1 + \lambda_1}, q_2\right) + \frac{1 - \lambda_2}{\lambda_1 - \lambda_2}\Pi\left(\frac{-2}{-\lambda_2 - 1}, q_2\right)\right] + \\ 2a^2\left(\frac{q_2}{2a(\rho\rho')^{1/2}}\right)\left[-(\lambda_1 + \lambda_2)K(q_2) + \frac{(\lambda_1 - 1)\lambda_1}{\lambda_1 - \lambda_2}\Pi\left(\frac{2}{1 + \lambda_1}, q_2\right) + \frac{(1 - \lambda_2)\lambda_2}{\lambda_1 - \lambda_2}\Pi\left(\frac{-2}{-\lambda_2 - 1}, q_2\right)\right] + \\ \left.\frac{1}{\rho\rho'}(a^2 - b^2)\pi + (\lambda_1 + \lambda_2)\pi - \frac{\pi}{\rho\rho'}(\lambda_1^2 - 1)^{1/2}[\rho\rho'\lambda_1 + z_F^2] - \frac{\pi}{\rho\rho'}(\lambda_2^2 - 1)^{1/2}[\rho\rho'\lambda_2 + z_F^2]\right\} \end{aligned} \tag{A7}$$

When ρ or ρ' is zero,

$$H_{0i}(0, \rho') = H_{0i}(\rho, 0) = 0 \qquad i = 1, 2, 3 \tag{A8}$$

It was found that there is a precision problem in Eq. (A7) when ρ and ρ' are near zero. This might be anticipated, as

there is a very small denominator proportional to $(\rho\rho')^{3/2}$, and the result is expected to be small in view of Eq. (A8). After somewhat laborious manipulations with Eq. (A7) to resolve the precision problem, H_{03} is obtained in the following form (with $b = 1$):

$$H_{03} = \frac{1}{2\pi a}\left\{\frac{1}{\rho\rho' Q_s} q_2 D(q_2) + \frac{1}{\rho\rho' R_{th}}(R_{th}^2 - 1)P(q_2) - \frac{1}{\rho\rho'} Q_s[\rho^2\rho'^2 + z_F^2[\rho^2 + \rho'^2)]\cdot P(q_2) - \frac{2Q_s}{\rho\rho'}[\rho^2\rho'^2 + z_F^2(\rho^2 + \rho'^2) + (b^4 - a^2b^2)]\left[\frac{\lambda_1 - 1}{\lambda_1 - \lambda_2}\Pi\left(\frac{2}{1+\lambda_1}, q_2\right) + S_e\right] + 2a^2Q_s\left[\frac{(1-\lambda_1)(1+\lambda_1)}{\lambda_1 - \lambda_2}P(q_2) + \frac{(\lambda_1 - 1)\lambda_1}{\lambda_1 - \lambda_2}\times \Pi\left(\frac{2}{1+\lambda_1}, q_2\right) + \frac{(1-\lambda_2^2)}{\lambda_1 - \lambda_2}\Pi_N + R_{H3}\right]\right\} \quad \text{(A9)}$$

where Q_s, R_{th}, R_{H3}, R_1, R_2, R_3, R_{ta}, R_4, $(R_{th}^2 - 1)$ are algebraic expressions given in Refs. 1 and 2 and

$$S_e = -\frac{(\lambda_1 - 1)}{\lambda_1 - \lambda_2}\left[\frac{1}{\lambda_2}K(q_2) + P(q_2)\right] + \frac{K(q_2)}{\lambda_2} - \frac{(\lambda_2^2 - 1)}{(\lambda_1 - \lambda_2)\lambda_2}\Pi_N \quad \text{(A10)}$$

It is recalled that Budiansky's technique of differentiation under the integral sign does not seem to lead to simple results because of the presence of nonzero z_F, the relative depth measured from the center of the spherical tank.

Appendix B: Analytic Expression for $F(\rho, \rho')$

Using the same technique as in Appendix A, one finds

$$F_0(\bar{\rho}, \bar{\rho}') = \int_0^{2\pi}\left.\frac{\partial G_0}{\partial z'}\right|_{z=z'=z_F}\cos\sigma\, d\sigma = F_1 + F_2 + F_3 \quad \text{(B1)}$$

where

$$F_1(\rho, \rho') = \left[\frac{E(q_2)}{(1 - q_2^2)} + 2D(q_2)\right]\frac{Q_s^3 z_F a}{\pi}(b + \rho)(b - \rho) \quad \text{(B2)}$$

$$F_2 = \frac{-z_F}{2\rho\rho' a}\times \left\{\frac{1}{2}\left[\frac{2\rho^2\rho'^2 + (z_F^2 + R_{ta})[(\rho^2 + \rho'^2) - |\rho^2 - \rho'^2|]}{|z_F|R_{ti}}\right]\cdot \frac{1}{R_{ta}}\cdot\frac{z_F^2 + \rho^2}{z_F^2 + \rho^2 + R_{ta}}\cdot\frac{(\rho - \rho')}{|(\rho - \rho')|} - \frac{\rho^2\rho'^2}{|z_F|R_{ti}[|z_F|R_{ti} + z_F^2 + R_{ta}]}\right\} \quad \text{(B3)}$$

R_{ta}, R_{ij} are algebraic expressions given in Refs. 1 and 2. Finally, F_3 will be expressed in closed form as follows:

$$F_3(\rho, \rho') = -\frac{z_F}{4\pi}\int_0^{2\pi}\frac{1}{arR'r^2r'^2(\cos^2\gamma - 1)}[(a^2r^2 + r^2r'^2) - (r^2 + a^2)rr'\cos\gamma]\cdot\cos\sigma d\sigma = \frac{z_F q_2}{2\pi a^2(\rho\rho')^{3/2}}\times \left\{(z_F^2 + \rho^2 + a^2)K(q_2) + \frac{1}{2}\cdot \left[1 - \frac{(\rho\rho' - z_F^2)[z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2\rho'^2]^{1/2}}{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2\rho'^2}\right]\cdot$$

(Equation continued in next column.)

$$\frac{1}{z_F^2(\rho + \rho')^2}\cdot[\{-z_F^2 + [z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2\rho'^2]^{1/2}\}\times [2z_F^4 + z_F^2(2a^2 + 2\rho^2 + \rho'^2) + \rho^2(a^2 + \rho'^2)] - (z_F^2 + a^2 + \rho^2)[z_F^2(\rho^2 + \rho'^2) + \rho^2\rho'^2]]\cdot\Pi\left(\frac{2}{1+\lambda_1}, q_2\right) - \frac{1}{2}\left[1 + \frac{(\rho\rho' - z_F^2)[z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2\rho'^2]^{1/2}}{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2\rho'^2}\right]\cdot \frac{1}{z_F^2(\rho + \rho')^2}\cdot\{(z_F^2 + [z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2\rho'^2)]^{1/2}\cdot [2z_F^4 + z_F^2(2a^2 + 2\rho^2 + \rho'^2) + \rho^2(a^2 + \rho'^2)] + (z_F^2 + \rho^2 + a^2)[z_F^2(\rho^2 + \rho'^2) + \rho^2\rho'^2]\}\Pi\left(\frac{-2}{-\lambda_2 - 1}, q_2\right)\right\} \quad \text{(B4)}$$

There is a serious precision problem for ρ, ρ' small in Eq. (B4). After manipulations, the precision problem is resolved by employing the following equivalent form:

$$F_3 = \frac{z_F q_2}{2\pi a^2(\rho\rho')^{3/2}}\{F_{31} + F_{32}\} \quad \text{(B5)}$$

where

$$F_{31} = -\{(z_F^2 + \rho^2 + a^2)\Pi_n + \tfrac{1}{2}[B_2 + 2(z_F^2 + \rho^2 + a^2)\cdot B_1 + B_1B_2][K(q_2) + \Pi_n]\} \quad \text{(B6)}$$

$$F_{32} = C_1^*C_2^*C_3^*\Pi\left(\frac{2}{1+\lambda_1}, q_2\right) \quad \text{(B7)}$$

where B_1, B_2, C_1^*, C_2^*, C_3^* are algebraic expressions given in Refs. 1 and 2. When ρ or $\rho' \to 0$, $F_i \to 0$ for $i = 1, 2, 3$. It is important to note that, whether $\rho > \rho'$ or $\rho' > \rho$, the sum of F_1, F_2, and F_3 always approaches zero as $z_F \to 0$. Therefore, for a half-full tank $H = H_0$, which is in agreement with Budiansky's kernel function aside from an apparent factor of two difference mentioned previously under Eq. (19).

Appendix C: Subroutines *DKEF* and *NEFF* (WIZ Program) [*DKEF* = *DKEF* (k, k', 1, 1, 1), *NEFF* = *NEFF* (β, κ, κ', 1, 1)]

The unity arguments are actually dummies, whereas the five arguments represent five outputs. For *DKEF*, the outputs are $K(k)$ = *DKEF* (1), $E(k)$ = *DKEF* (2), $[E(k) - K(k)]/k^2$ = *DKEF* (3), $K(k) - \pi/2$ = *DKEF* (4), and the number of iterations = *DKEF* (5). *DKEF* (3) is not obtained from *DKEF* (1) − *DKEF* (2) but is obtained after a significant analytic subtraction in the program. For *NEFF*, the outputs are $F(\beta, k = \kappa)$ = *NEFF* (1), $E(\beta, k = \kappa)$ = *NEFF* (2), $[E(\beta, k) - k\sin\beta]$ = *NEFF* (3); the number of iterations for evaluating $F(\beta, k)$ = *NEFF* (4); the number of iterations for evaluating *NEFF* (3) = *NEFF* (5). *NEFF* (3) is evaluated after a significant analytic subtraction in the program, whereas *NEFF* (2) is simply obtained from *NEFF* (3) + $k\sin\beta$. Although $k' = (1 - k^2)^{1/2}$ does not appear in the functions sought, it is calculated from a formula without subtraction, as one can easily see that significant figures of k' would be lost if k is near unity. The basic formulas are all given in Ref. 17. For complete elliptic integrals, the iterative method based on geometric and arithmetic means was employed. For incomplete elliptical integrals, the iterative method based on inverse order of transformation was employed in order to construct *NEFF* (3). The programs are written in "WIZ" language for GE-225 computers, which is analogous to "FORTRAN" for IBM computers, and are given in Refs. 1 and 2.

Appendix D: Derivation of π_N

For $\Pi[-2/(-1 - \lambda_2), q_2]$, the formulas 410.01 and 411.01 of Ref. 16 are applicable in which $\alpha^2 = -2/(-1 - \lambda_1) <$

0, $k = q_2$. One can further apply the addition formula (153.01 of Ref. 16) restricted to the condition that $k \tan\beta \cdot \tan\psi = 1$. Eliminating $\Lambda_0(\beta, k) - 1$, one finds

$$\Pi_N = \left[\Pi_2 - \frac{K(k)}{1 - (\alpha^2/2)}\right] =$$

$$- \alpha^2 \left\{\frac{\pi}{2} \frac{[1 - (\alpha^2/2)]L_{id}}{[-\alpha^2(1 - \alpha^2)(k^2 - \alpha^2)]^{1/2}} - \frac{(1/2)K(k)}{(1 - \alpha^2)}\right\} \quad \text{(D1)}$$

$$L_{id} = (1 - \Lambda_0) = [1 - \sin\beta - (\Lambda_0 - \sin\beta)] =$$

$$\frac{-\alpha^2}{(1 - \alpha^2)^{1/2}[1 + (1 - \alpha^2)^{1/2}]} - \frac{2}{\pi}\left\{ k^2 D(k) F(\beta, k') + \right.$$

$$\left. P(k)E(\beta, k') + \frac{\pi}{2} D(\beta, k') - \frac{\pi}{2} \frac{k^2}{(1 + k')} \frac{1}{(1 - \alpha^2)^{1/2}}\right\} \quad \text{(D2)}$$

There is apparently a gain of significant figures of Π_N when ρ_i, ρ_j are small (α, k small) if Eq. (A7) is used, provided that the first term in the brackets can be evaluated as accurately as the second term. This is achieved by employing the subroutines *DKEF* and *NEFF* for Eq. (D2).

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